

# Small-amplitude capillary-gravity water waves: exact solutions and particle motion beneath such waves

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## Abstract

Two-dimensional periodic surface waves propagating under the combined influence of gravity and surface tension on water of finite depth are considered. Within the framework of small-amplitude waves, we find the exact solutions of the nonlinear differential equation system which describes the particle motion in the considered case, and we describe the possible particle trajectories. The required computations involve elliptic integrals of the first kind, the Legendre normal form and a solvable Abel differential equation of the second kind. Some graphs of the results are included.

## 1 Introduction

We are concerned with the motion of periodic plane waves which are propagated in water of finite depth and with free surface, under a gravitational field. The governing equations of motion are the incompressible Euler equations. On the free surface we take into account the influence of surface tension. The surface tension will appear in the formulation of the boundary conditions. Thus, one obtains the boundary-value problem for capillary-gravity waves. There are very few explicit solutions known for the water-wave problems. The first such solution for pure gravity water waves, was described by Gerstner [19]. This solution was independently rediscovered later by Rankine [37]. Modern detailed descriptions of this wave are given in recent papers [2] and [23]. Gerstner's solution is restrictive: it exists only for deep water and it describes a rotational wave. Related to Gerstner's solution, Constantin, in [3], constructed an explicit rotational solution to the nonlinear governing equations describing gravity water waves that progress along the shoreline. Beneath Gerstner's waves it is possible to have a motion of the fluid where all particles describe circles with a depth-dependent radius ([2], [23]). Crapper [14] derived an exact solution for pure

capillary waves travelling at a constant velocity at the surface of a fluid of infinite depth. He showed that as the amplitude increases, the waves develop broad crests and sharp troughs. Crapper's solutions ultimately reach a limiting form for which there is a trapped bubble at the troughs. For higher values of the amplitude the free-surface profiles are self-intersecting and therefore non physical. In [24], by the use of the general method for calculating trajectories given by Longuet-Higgins [34], the particle trajectories in Crapper's nonlinear capillary waves are derived. It is found that the orbits of the steeper waves are neither circular nor closed. Crapper's solution was extended in the case of a fluid of finite depth by Kinnersley [32]. The particle trajectories for Kinnersley's waves were calculated in [26]. For nonlinear capillary-gravity waves no exact analytic solution has yet been found. Making use of numerical studies, in [25] the particle trajectories in irrotational nonlinear capillary-gravity waves are investigated. There are several unanswered questions about water waves with surface tension and vorticity. The existence of regular periodic travelling waves with vorticity was recently established (see [11], [42]). For steady periodic gravity waves the symmetry is known to be ubiquitous (see [6], [27]). The study of the symmetry of rotational water waves was initiated in [7], [8]; for irrotational flows see also [36]. However, exact information about the flow beneath such waves is not readily available, even in the irrotational case. This paper addresses this issue.

It is fortunate that a great number of observations can be explained on the basis of the small-amplitude wave theory. Within this framework, in this paper we investigate the capillary-gravity waves and the internal motion of the fluid under the passage of such waves. We simplify the full system of equations by a linearization found in [28], which is around still water and which is slightly different from the classical case in line with the Stokes condition for irrotational flows (for the latter, see for example, [5] and [10] where perturbations of laminar flows are considered, that is, flows characterized by a flat surface  $\eta = 0$ ; these two approaches to linearization are surveyed in [18]). After rewriting the governing equations of motion and the boundary conditions for the capillary-gravity water-wave problem in an appropriate non-dimensional form, three non-dimensional parameters arise:  $\epsilon$  = amplitude/depth, the amplitude parameter,  $\delta$  = depth/wavelength, the shallowness parameter and  $W_e$ , called Weber number, which comes from the surface tension on the free surface. These parameters help us to characterize the various types of approximation. We suppose that the water flow is irrotational; thus, in addition to the full system of equations we also have the irrotational condition that we write in the suitable set of non-dimensional variables. The linearized problem is obtained by letting  $\epsilon \rightarrow 0$ ,  $\delta$  and  $W_e$  being fixed. Solving this problem, we obtain a parameter  $c_0$  by which we can describe different backward flows in the irrotational case: still water ( $c_0 = 0$ ), favorable uniform current  $c_0 > 0$  and adverse uniform current  $c_0 < 0$ . After getting the general solution of the linearized water-wave problem we investigate further the nonlinear equations of the motion of the fluid particles to offer much insight into the fluid motion.

It was widely believed that as the small-amplitude waves propagate on the surface the particles of the fluid move on closed orbits. Analyzing the first-order

approximation of the nonlinear ordinary differential equation system which describes the particle motion, it was indeed obtained that all water particles trace closed, circular or elliptic, orbits (see, for example, [15], [31], [33], [35], [39], [40]). But in [10] it was proved, using phase-plane considerations for the nonlinear ordinary differential equation system which describes the particle motion, that in linear irrotational gravity water waves no particle trajectory is actually closed, unless the free surface is flat. Each particle trajectory involves over a period a backward/forward movement, and the path is an elliptical arc with a forward drift; on the flat bed the particle path degenerates to a backward/forward motion. In [18] recent results in the understanding of particle paths within different types of progressive water waves are surveyed, in the framework of linear theory as well as in the framework of exact theory of periodic symmetric waves, and in the presence or not of the background currents and vorticity. The results are in agreement with Stokes' observation for a net mass drift [41]. In the linear framework, by using phase-plane considerations, one obtains that the particle trajectories in linear deep-water waves (see [5]), in linear gravity water waves over finite depth with constant vorticity (see [16], [17]), and in linear irrotational capillary and capillary-gravity water waves (see [21], [22]) are not closed. Small-amplitude shallow-water waves were studied in [28] (the irrotational case) and [29] (the constant vorticity case), and the exact solutions of the nonlinear ordinary differential equation system which describes the particle motion below such waves were found. Depending on the strength of the underlying uniform current [28] or depending on the relation between the initial data and the constant vorticity [29], it was obtained that some particle trajectories are undulating curves to the right, or to the left, others are loops with forward drift, or with backward drift, and others can follow some peculiar shapes. In the framework of exact theory, the forward drift of the particles as the wave progresses was given by analyzing a free-boundary problem for harmonic functions in a planar domain (see [4] for Stokes waves, [9] for solitary waves and [20] for deep-water Stokes waves) or by applying local bifurcation theory (see [43] for small-amplitude waves with vorticity).

In this paper we continue the study started in [30]. We provide explicit solutions for the ordinary differential equation system describing the motion of the particles beneath small-amplitude capillary-gravity waves which propagate on the surface of an irrotational water flow with a flat bottom. In the case when the constant  $c_0$  equals the non-dimensional speed of propagation of the linear wave, the required computations involve elliptic integrals of the first kind and their Legendre's normal form. The six exact solutions obtained in this case contain Jacobian elliptic functions in their expressions. In the case when the constant  $c_0$  is different from the non-dimensional speed of propagation of the linear wave, the computations involve a solvable Abel differential equation of the second kind. In both cases we remark that the solutions obtained are not closed curves. We analyze some solutions in detail and we draw their graphs.

## 2 Preliminaries

### 2.1 The water-wave problem

For two-dimensional periodic waves the motion is identical in any direction parallel to the crest line and is periodic in the spatial direction in which the wave is propagating. To describe these waves we consider a cross section of the flow that is perpendicular to the crest line with Cartesian coordinates  $(x, z)$ , the  $x$ -axis being in the direction of wave propagation and the  $z$ -axis pointing vertically upwards. The water flow under consideration is bounded by a rigid horizontal surface below at  $z = 0$  and a free surface above at  $z = h_0 + \eta(x, t)$ , where the constant  $h_0 > 0$  is the mean water level. We denote by  $(u(x, z, t), v(x, z, t))$  the velocity of the water. Let gravity act now in concert with surface tension. The surface tension will play a role in the formulation of the boundary conditions but not in the equations of motion valid in the fluid domain. Assuming that the water is both homogeneous (constant density  $\rho$ ) (see [35]) and inviscid, we obtain within the fluid domain the equation of mass conservation (MC) together with Euler's equations (EEs) (see [31]). The boundary conditions for the water-wave problem are the kinematic boundary conditions as well as the dynamic boundary condition. The kinematic boundary conditions (KBCs) express the fact that the same particles always form the free-water surface and that the fluid is assumed to be bounded below by a hard horizontal bed  $z = 0$ . The dynamic boundary condition (DBC) expresses the fact that the difference of pressure on the two sides of the surface  $\eta$  is balanced by the effects of surface tension. Thus, the boundary-value problem for capillary-gravity water waves is

$$\begin{aligned}
 u_t + uu_x + vv_z &= -\frac{1}{\rho}p_x & (\text{EE}) \\
 v_t + uv_x + vv_z &= -\frac{1}{\rho}p_z - g \\
 u_x + v_z &= 0 & (\text{MC}) \\
 v &= \eta_t + u\eta_x \text{ on } z = h_0 + \eta(x, t) & (\text{KBC}) \\
 v &= 0 \text{ on } z = 0 \\
 p &= p_0 - \frac{\Gamma}{R}, \text{ on } z = h_0 + \eta(x, t) & (\text{DBC})
 \end{aligned} \tag{1}$$

where  $p(x, z, t)$  is the pressure,  $g$  is the constant gravitational acceleration,  $p_0$  is the constant atmospheric pressure, the parameter  $\Gamma(> 0)$  is the coefficient of surface tension and  $\frac{1}{R}$  is the mean curvature (up to a factor  $1/2$ ) of the surface. For a surface defined as the function  $\eta(x, t)$ , the mean curvature has the following expression

$$\frac{1}{R} = \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \tag{2}$$

In respect of the well-posedness for the initial-value problem for (1) there has been significant recent progress; see [13] and the references therein.

An important category of flows is those of zero vorticity (irrotational flows), characterized by the additional equation

$$u_z - v_x = 0 \tag{3}$$

In what follows we will consider this type of flow. The idealization of irrotational flow is physically relevant in the absence of non-uniform currents in the water.

## 2.2 Small-amplitude approximation of the water-wave problem

We search for a linear approximation of the water-wave problem (1)-(3). We set the constant water density  $\rho = 1$ . If  $\lambda > 0$  is the wavelength and  $a > 0$  is the wave amplitude, we make the following change of variables (yielding the non-dimensionalization of the problem; see [31])

$$\begin{aligned} x &\mapsto \lambda x, & z &\mapsto h_0 z, & \eta &\mapsto a\eta, & t &\mapsto \frac{\lambda}{\sqrt{gh_0}} t, \\ u &\mapsto \sqrt{gh_0} u, & v &\mapsto h_0 \frac{\sqrt{gh_0}}{\lambda} v \end{aligned} \quad (4)$$

$$p \mapsto p_0 + gh_0(1 - z) + gh_0 p \quad (5)$$

where, to avoid new notation, we have used the same symbols for the non-dimensional variables  $x, z, \eta, t, u, v, p$  on the right-hand side. The non-dimensional pressure variable measures the deviation from the hydrostatic pressure  $p_0 + gh_0(1 - z)$ . We now apply the following scaling of the non-dimensional variables

$$p \mapsto \epsilon p, \quad (u, v) \mapsto \epsilon(u, v) \quad (6)$$

avoiding again the introduction of new notation.

Taking into account (4), (5) and (6), the two-dimensional capillary-gravity waves on irrotational water of finite depth are described, in non-dimensional scaled variables, by the following boundary-value problem:

$$\begin{aligned} u_t + \epsilon(uu_x + vv_z) &= -p_x \\ \delta^2[v_t + \epsilon(uv_x + vv_z)] &= -p_z \\ u_x + v_z &= 0 \\ u_z - \delta^2 v_x &= 0 \\ v &= \eta_t + \epsilon u \eta_x && \text{on } z = 1 + \epsilon \eta(x, t) \\ p &= \eta - \left( \frac{\Gamma}{g\lambda^2} \right) \frac{\eta_{xx}}{(1 + \epsilon^2 \delta^2 \eta_x^2)^{3/2}} && \text{on } z = 1 + \epsilon \eta(x, t) \\ v &= 0 && \text{on } z = 0 \end{aligned} \quad (7)$$

where we have introduced the amplitude parameter  $\epsilon = \frac{a}{h_0}$  and the shallowness parameter  $\delta = \frac{h_0}{\lambda}$ . It is conventional to write  $\frac{\Gamma}{\rho g \lambda^2} = \delta^2 W_e$ , with  $W_e = \frac{\Gamma}{\rho g h_0^2}$  a Weber number. This parameter is used to measure the size of the surface tension contribution.

By letting  $\epsilon \rightarrow 0$ ,  $\delta$  and  $W_e$  being fixed, we obtain a linear approximation of

our problem, that is,

$$\begin{aligned}
 u_t + p_x &= 0 \\
 \delta^2 v_t + p_z &= 0 \\
 u_x + v_z &= 0 \\
 u_z - \delta^2 v_x &= 0 \\
 v &= \eta_t && \text{on } z = 1 \\
 p &= \eta - \delta^2 W_e \eta_{xx} && \text{on } z = 1 \\
 v &= 0 && \text{on } z = 0
 \end{aligned} \tag{8}$$

Manipulating the first four equations of system (8), we obtain that

$$v_{zz} + \delta^2 v_{xx} = 0 \tag{9}$$

Applying the method of separation of variables, we seek the solution of the equation (9) in the form

$$v(x, z, t) = F(x, t)G(z, t) \tag{10}$$

Substituting (10) into the equation (9), separating the variables and taking into account the expressions of  $v$  on the boundaries, that is, the fifth equation and the last equation in (8), we find

$$v(x, z, t) = \frac{1}{\sinh(k\delta)} \sinh(k\delta z) \eta_t \tag{11}$$

where  $k \geq 0$  is a constant that might depend on time. For the component  $u$  of the velocity field, taking into account (11) and the fourth equation of system (8), we obtain

$$u(x, z, t) = \frac{\delta}{k \sinh(k\delta)} \cosh(k\delta z) \eta_{tx} + \mathcal{F}(x, t) \tag{12}$$

where  $\mathcal{F}(x, t)$  is an arbitrary function. The components  $u$  and  $v$  of the velocity have to fulfill also the third equation in (8), hence, in view of (11) and (12),

$$\frac{\delta}{k \sinh(k\delta)} \cosh(k\delta z) \eta_{txx} + \frac{\partial \mathcal{F}(x, t)}{\partial x} = -\frac{k\delta}{\sinh(k\delta)} \cosh(k\delta z) \eta_t \tag{13}$$

The above relation must hold for all values of  $x \in \mathbf{R}$ , and  $0 \leq z \leq 1$ . It follows that the function  $\mathcal{F}(x, t)$  is independent of  $x$ , therefore we will denote this function by  $\mathcal{F}(t)$ , and

$$\eta_{txx} + k^2 \eta_t = 0 \tag{14}$$

We seek periodic travelling wave solutions; thus, for the equation (14) with

$$k = 2\pi \tag{15}$$

we choose the following solution

$$\eta(x, t) = \cos(2\pi(x - ct)) \tag{16}$$

where  $c$  represents the non-dimensional speed of propagation of the linear wave and is to be determined.

In order to find the expressions of the pressure we take into account the first two equations in (8) and the expressions of the velocity field from above. Thus, we obtain

$$p(x, z, t) = \frac{2\pi\delta c^2}{\sinh(2\pi\delta)} \cosh(2\pi\delta z) \cos(2\pi(x - ct)) + x\mathcal{F}'(t) \quad (17)$$

On the free surface  $z = 1$  the pressure (17) has to fulfill the sixth equation of system (8). Hence, in view of (16), we get

$$2\pi\delta c^2 \coth(2\pi\delta) \cos(2\pi(x - ct)) + x\mathcal{F}'(t) = (1 + 4\pi^2\delta^2 W_e) \cos(2\pi(x - ct)) \quad (18)$$

The above relation must hold for all values  $x \in \mathbf{R}$ ; therefore, we get

$$\mathcal{F}(t) = \text{constant} := c_0 \quad (19)$$

and we provide the non-dimensional speed of the linear wave

$$c^2 = \frac{\tanh(2\pi\delta)}{2\pi\delta} (1 + 4\pi^2\delta^2 W_e) = \frac{\lambda}{2\pi h_0} \left( 1 + \frac{4\pi^2\Gamma}{g\lambda^2} \right) \tanh\left(\frac{2\pi h_0}{\lambda}\right) \quad (20)$$

We observe thus, that the speed of propagation of the wave varies with the wavelength  $\lambda$ , with the undisturbed depth  $h_0$  and with the coefficient of surface tension  $\Gamma$ .

Summing up, system (8) has the solution

$$\begin{aligned} \eta(x, t) &= \cos(2\pi(x - ct)) \\ u(x, z, t) &= \frac{2\pi\delta c}{\sinh(2\pi\delta)} \cosh(2\pi\delta z) \cos(2\pi(x - ct)) + c_0 \\ v(x, z, t) &= \frac{2\pi c}{\sinh(2\pi\delta)} \sinh(2\pi\delta z) \sin(2\pi(x - ct)) \\ p(x, z, t) &= \frac{2\pi\delta c^2}{\sinh(2\pi\delta)} \cosh(2\pi\delta z) \cos(2\pi(x - ct)) \end{aligned} \quad (21)$$

with  $c$  given by (20).

### 3 Particle trajectories

Let  $(x(t), z(t))$  be the path of a particle in the fluid domain, with location  $(x(0), z(0)) := (x_0, z_0)$  at time  $t = 0$ . Below small-amplitude capillary-gravity water waves, taking into account (21), the motion of the particles is described by the following differential system

$$\begin{cases} \frac{dx}{dt} = u(x, z, t) = \frac{2\pi\delta c}{\sinh(2\pi\delta)} \cosh(2\pi\delta z) \cos(2\pi(x - ct)) + c_0 \\ \frac{dz}{dt} = v(x, z, t) = \frac{2\pi c}{\sinh(2\pi\delta)} \sinh(2\pi\delta z) \sin(2\pi(x - ct)) \end{cases} \quad (22)$$

The right-hand side of the differential system (22) is smooth and bounded; therefore, the unique solution of the Cauchy problem with initial data  $(x_0, z_0)$

is defined globally in time.

Notice that the constant  $c_0$  is the average of the horizontal fluid velocity over any horizontal segment of length 1; that is,

$$c_0 = \frac{1}{1} \int_x^{x+1} u(s, z, t) ds, \quad (23)$$

representing therefore the strength of the underlying uniform current (see also [12]). Thus,  $c_0 = 0$  will correspond to a region of still water with no underlying current,  $c_0 > 0$  will characterize a favorable uniform current and  $c_0 < 0$  will characterize an adverse uniform current.

To study the exact solution of system (22) it is more convenient to rewrite it in the following moving frame

$$X = 2\pi(x - ct), \quad Z = 2\pi\delta z \quad (24)$$

This transformation yields

$$\begin{cases} \frac{dX}{dt} = \frac{4\pi^2\delta c}{\sinh(2\pi\delta)} \cosh(Z) \cos(X) + 2\pi(c_0 - c) \\ \frac{dZ}{dt} = \frac{4\pi^2\delta c}{\sinh(2\pi\delta)} \sinh(Z) \sin(X) \end{cases} \quad (25)$$

#### I) $c_0 = c$

In this case, differentiating with respect to  $t$ , system (25) can be written into the following form:

$$\begin{cases} \frac{d^2 X}{dt^2} = -\frac{8\pi^4\delta^2 c^2}{\sinh^2(2\pi\delta)} \sin(2X) \\ \frac{d^2 Z}{dt^2} = \frac{8\pi^4\delta^2 c^2}{\sinh^2(2\pi\delta)} \sinh(2Z) \end{cases} \quad (26)$$

We denote by

$$A^2 := \frac{8\pi^4\delta^2 c^2}{\sinh^2(2\pi\delta)} \quad (27)$$

We observe that  $A^2$  as a function of  $\delta$  is decreasing and  $\lim_{\delta \rightarrow 0} A^2 = 2\pi^2 c^2$ ,  $\lim_{\delta \rightarrow \infty} A^2 = 0$ . Thus,

$$0 \leq A^2 \leq 2\pi^2 c^2 \quad (28)$$

System (26) integrates at

$$\begin{cases} \left(\frac{dX}{dt}\right)^2 = A^2 \cos(2X) + c_1 \\ \left(\frac{dZ}{dt}\right)^2 = A^2 \cosh(2Z) + c_2 \end{cases} \quad (29)$$

$c_1, c_2$  being the integration constants. Because the right-hand side of the first equation in (29) has to be bigger than zero, the constant  $c_1$  has to satisfy the following condition:

$$c_1 + A^2 > 0 \quad (30)$$

For the first equation in (29) we use the substitution

$$\tan(X) = y, \quad \cos(2X) = \frac{1 - y^2}{1 + y^2}, \quad \sin(2X) = \frac{2y}{1 + y^2}, \quad dX = \frac{1}{1 + y^2} dy \quad (31)$$



In the new variable, the first equation in (29) takes the form

$$\left(\frac{dy}{dt}\right)^2 = A^2(1 - y^4) + c_1(1 + y^2)^2 \quad (32)$$

The solution of the equation (32) involves an elliptic integral of the first kind:

$$\pm \int \frac{dy}{\sqrt{(c_1 - A^2)y^4 + 2c_1y^2 + c_1 + A^2}} = t \quad (33)$$

The elliptic integral of the first kind from (33) may be reduced to the Legendre normal form. In order to do this we consider the substitution

$$y^2 = s \quad (34)$$

Therefore, the left-hand side in (33) becomes

$$\pm \int \frac{dy}{\sqrt{(c_1 - A^2)y^4 + 2c_1y^2 + c_1 + A^2}} = \pm \int \frac{ds}{2\sqrt{(c_1 - A^2)s(s+1)\left(s + \frac{c_1 + A^2}{c_1 - A^2}\right)}} \quad (35)$$

Further, we introduce a new variable  $\varphi$ . The definition of this variable depends on the sign of  $c_1 - A^2$ .

If

$$c_1 - A^2 > 0 \quad (36)$$

then we introduce the variable  $\varphi$  by

$$s = \tan^2 \varphi \quad (37)$$

and we get

$$(c_1 - A^2)s(s+1) \left(s + \frac{c_1 + A^2}{c_1 - A^2}\right) = (c_1 + A^2) \frac{\tan^2 \varphi}{\cos^4 \varphi} [1 - k_1^2 \sin^2 \varphi]$$

$$ds = 2 \frac{\tan \varphi}{\cos^2 \varphi} d\varphi$$

where the constant  $0 < k_1^2 < 1$  is given by

$$k_1^2 = \frac{2A^2}{c_1 + A^2} \quad (38)$$

Therefore we obtain the Legendre normal form of the integral in (33); that is,

$$\pm \frac{1}{\sqrt{c_1 + A^2}} \int \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}} = t \quad (39)$$

The inverse of the integral in (39) is the Jacobian elliptic function *sine amplitude* (see, for example, [1]), an odd periodic function of order two,

$$\operatorname{sn} \left( \pm \sqrt{c_1 + A^2} t; k_1 \right) := \sin \varphi \quad (40)$$

In view of the notations (34), (37), we get that

$$y(t) = \pm \frac{\operatorname{sn}(\sqrt{c_1 + A^2} t; k_1)}{\operatorname{cn}(\sqrt{c_1 + A^2} t; k_1)} := \pm \operatorname{sc}(\sqrt{c_1 + A^2} t; k_1) \quad (41)$$

where  $\operatorname{cn}(\sqrt{c_1 + A^2} t; k_1) := \cos \varphi$  is the Jacobian elliptic function *cosine amplitude*, an even periodic function of order two, and  $\operatorname{sc}$  is Glaisher's notation for the quotient  $\operatorname{sn}/\operatorname{cn}$  (see, for example, [1]).

If

$$c_1 - A^2 < 0 \quad (42)$$

then, taking also into account the condition (30), we introduce the variable  $\varphi$  by (see [38] Ch. VI, §4, page 602)

$$s = \frac{A^2 + c_1}{A^2 - c_1} \cos^2 \varphi \quad (43)$$

In this case we get (see [30])

$$y(t) = \pm \sqrt{\frac{A^2 + c_1}{A^2 - c_1}} \operatorname{cn}(\sqrt{2A^2} t; k_2) \quad (44)$$

where the constant  $0 < k_2^2 < 1$  is given by

$$k_2^2 = \frac{A^2 + c_1}{2A^2} \quad (45)$$

For the second equation in (29) we use the substitution

$$\tanh(Z) = w, \quad \cosh(2Z) = \frac{1 + w^2}{1 - w^2}, \quad dZ = \frac{1}{1 - w^2} dw \quad (46)$$

In the new variable, the second equation in (29) takes the form

$$\left(\frac{dw}{dt}\right)^2 = A^2(1 - w^4) + c_2(1 - w^2)^2 \quad (47)$$

The solution of the equation (47) involves an elliptic integral of the first kind:

$$\pm \int \frac{dw}{\sqrt{(c_2 - A^2)w^4 - 2c_2w^2 + c_2 + A^2}} = t \quad (48)$$

The elliptic integral of the first kind from (48) may be reduced to the Legendre normal form. In order to do this we consider the substitution

$$w^2 = r \quad (49)$$

The left-hand side in (48) becomes

$$\pm \int \frac{dw}{2\sqrt{(c_2 - A^2)w^4 - 2c_2w^2 + c_2 + A^2}} = \pm \int \frac{dr}{2\sqrt{(c_2 - A^2)r(r-1)(r - \frac{c_2 + A^2}{c_2 - A^2})}} \quad (50)$$

As in the case of the integral in (35), we introduce a new variable  $\phi$ . The definition of  $\phi$  depends on the sign of  $c_2 - A^2$  and  $c_2 + A^2$ . There are three possibilities:

$$\begin{aligned} c_2 - A^2 &> 0, \\ c_2 - A^2 &< 0 \text{ and } c_2 + A^2 > 0, \\ c_2 + A^2 &< 0. \end{aligned}$$

If

$$c_2 - A^2 > 0 \quad (51)$$

then we introduce the variable  $\phi$  by (see [38] Ch. VI, §4, page 602)

$$r = \sin^2 \phi \quad (52)$$

and we get

$$\begin{aligned} (c_2 - A^2)r(r-1) \left( r - \frac{c_2 + A^2}{c_2 - A^2} \right) &= (c_2 + A^2) \sin^2 \phi \cos^2 \phi (1 - k_3^2 \sin^2 \phi) \\ dr &= 2 \sin \phi \cos \phi d\phi \end{aligned}$$

where the constant  $0 < k_3^2 < 1$  is given by

$$k_3^2 = \frac{c_2 - A^2}{c_2 + A^2} \quad (53)$$

Therefore we obtain the Legendre normal form of the integral in (48); that is,

$$\pm \frac{1}{\sqrt{c_2 + A^2}} \int \frac{d\phi}{\sqrt{1 - k_3^2 \sin^2 \phi}} = t \quad (54)$$

The inverse of the integral in (54) is the Jacobian elliptic function  $\text{sn}$

$$\text{sn} \left( \pm \sqrt{c_2 + A^2} t; k_3 \right) := \sin \phi \quad (55)$$

In view of the notations (49), (52), we get that

$$w(t) = \pm \text{sn} \left( \sqrt{c_2 + A^2} t; k_3 \right) \quad (56)$$

If

$$c_2 - A^2 < 0 \quad \text{and} \quad c_2 + A^2 > 0 \quad (57)$$

then we introduce the variable  $\phi$  by (see [38] Ch. VI, §4, page 602)

$$r = \cos^2 \phi \quad (58)$$

and we get

$$\begin{aligned} (c_2 - A^2)r(r-1) \left( r - \frac{c_2 + A^2}{c_2 - A^2} \right) &= 2A^2 \sin^2 \phi \cos^2 \phi (1 - k_4^2 \sin^2 \phi) \\ dr &= -2 \sin \phi \cos \phi d\phi \end{aligned}$$

where the constant  $0 < k_4^2 < 1$  is given by

$$k_4^2 = \frac{A^2 - c_2}{2A^2} \quad (59)$$

Therefore we obtain the Legendre normal form of the integral in (48); that is,

$$\pm \frac{1}{\sqrt{2}A} \int \frac{d\phi}{\sqrt{1 - k_4^2 \sin^2 \phi}} = t \quad (60)$$

The inverse of the integral in (60) is

$$\text{sn} \left( \pm \sqrt{2A^2} t; k_4 \right) := \sin \phi \quad (61)$$

In view of the notations (49), (58), we get that

$$w(t) = \pm \text{cn} \left( \sqrt{2A^2} t; k_4 \right) \quad (62)$$

If

$$c_2 + A^2 < 0 \quad (63)$$

then we introduce the variable  $\phi$  by (see [38] Ch. VI, §4, page 602)

$$r = 1 + \frac{2A^2}{c_2 - A^2} \sin^2 \phi \quad (64)$$

In this case we get (see [30])

$$w(t) = \pm \sqrt{1 - \frac{2A^2}{A^2 - c_2} \text{sn}^2 \left( \sqrt{A^2 - c_2} t; k_5 \right)} \quad (65)$$

where the constant  $0 < k_5^2 < 1$  is given by

$$k_5^2 = \frac{2A^2}{A^2 - c_2} \quad (66)$$

Thus, with (31) and (46) in view, the solution of system (29) has the following expression

$$\begin{aligned} X(t) &= \arctan [y(t)] \\ Z(t) &= \text{arctanh} [w(t)] = \frac{1}{2} \ln \frac{1+w(t)}{1-w(t)} \end{aligned} \quad (67)$$

with  $y(t)$  given by (41) or (44) and  $w(t)$  given by (56) or (62) or (65). From (24) and (67), the solutions of system (22) with the constant  $c_0$  equal to the speed of propagation of the linear wave  $c$  have the following expressions:

$$\begin{cases} x(t) = ct \pm \frac{1}{2\pi} \arctan \left[ \text{sc} \left( \sqrt{c_1 + A^2} t; \frac{2A^2}{c_1 + A^2} \right) \right] \\ z(t) = \pm \frac{1}{2\pi\delta} \text{arctanh} \left[ \text{sn} \left( \sqrt{c_2 + A^2} t; \frac{c_2 - A^2}{c_2 + A^2} \right) \right] \end{cases} \quad (68)$$

$$\begin{cases} x(t) = ct \pm \frac{1}{2\pi} \arctan \left[ \operatorname{sc} \left( \sqrt{c_1 + A^2} t; \frac{2A^2}{c_1 + A^2} \right) \right] \\ z(t) = \pm \frac{1}{2\pi\delta} \operatorname{arctanh} \left[ \operatorname{cn} \left( \sqrt{2A^2} t; \frac{A^2 - c_2}{2A^2} \right) \right] \end{cases} \quad (69)$$

$$\begin{cases} x(t) = ct \pm \frac{1}{2\pi} \arctan \left[ \operatorname{sc} \left( \sqrt{c_1 + A^2} t; \frac{2A^2}{c_1 + A^2} \right) \right] \\ z(t) = \pm \frac{1}{2\pi\delta} \operatorname{arctanh} \left[ \sqrt{1 - \frac{2A^2}{A^2 - c_2}} \operatorname{sn}^2 \left( \sqrt{A^2 - c_2} t; \frac{2A^2}{A^2 - c_2} \right) \right] \end{cases} \quad (70)$$

$$\begin{cases} x(t) = ct \pm \frac{1}{2\pi} \arctan \left[ \sqrt{\frac{A^2 + c_1}{A^2 - c_1}} \operatorname{cn} \left( \sqrt{2A^2} t; \frac{c_1 + A^2}{2A^2} \right) \right] \\ z(t) = \pm \frac{1}{2\pi\delta} \operatorname{arctanh} \left[ \operatorname{sn} \left( \sqrt{c_2 + A^2} t; \frac{c_2 - A^2}{c_2 + A^2} \right) \right] \end{cases} \quad (71)$$

$$\begin{cases} x(t) = ct \pm \frac{1}{2\pi} \arctan \left[ \sqrt{\frac{A^2 + c_1}{A^2 - c_1}} \operatorname{cn} \left( \sqrt{2A^2} t; \frac{c_1 + A^2}{2A^2} \right) \right] \\ z(t) = \pm \frac{1}{2\pi\delta} \operatorname{arctanh} \left[ \operatorname{cn} \left( \sqrt{2A^2} t; \frac{A^2 - c_2}{2A^2} \right) \right] \end{cases} \quad (72)$$

$$\begin{cases} x(t) = ct \pm \frac{1}{2\pi} \arctan \left[ \sqrt{\frac{A^2 + c_1}{A^2 - c_1}} \operatorname{cn} \left( \sqrt{2A} t; \frac{c_1 + A^2}{2A^2} \right) \right] \\ z(t) = \pm \frac{1}{2\pi\delta} \operatorname{arctanh} \left[ \sqrt{1 - \frac{2A^2}{A^2 - c_2}} \operatorname{sn}^2 \left( \sqrt{A^2 - c_2} t; \frac{2A^2}{A^2 - c_2} \right) \right] \end{cases} \quad (73)$$

Let us analyze in more detail the solution (68). Taking into account the expressions for the derivatives of sine amplitude and cosine amplitude (see, for example, [1]), that is,

$$\begin{aligned} \frac{d}{dt} \operatorname{sn}(t; k) &= \operatorname{cn}(t; k) \operatorname{dn}(t; k) \\ \frac{d}{dt} \operatorname{cn}(t; k) &= -\operatorname{sn}(t; k) \operatorname{dn}(t; k), \end{aligned} \quad (74)$$

where

$$\operatorname{dn}(t; k) := \sqrt{1 - k^2 \operatorname{sn}^2(t; k)}, \quad (75)$$

we get the derivatives with respect to  $t$  of  $x(t)$  and  $z(t)$  from (68):

$$\begin{cases} x'(t) = c \pm \frac{\sqrt{c_1 + A^2}}{2\pi} \operatorname{dn} \left( \sqrt{c_1 + A^2} t; \frac{2A^2}{c_1 + A^2} \right) \\ z'(t) = \pm \frac{\sqrt{c_2 + A^2}}{2\pi\delta} \frac{\operatorname{dn} \left( \sqrt{c_2 + A^2} t; \frac{c_2 - A^2}{c_2 + A^2} \right)}{\operatorname{cn} \left( \sqrt{c_2 + A^2} t; \frac{c_2 - A^2}{c_2 + A^2} \right)} \end{cases} \quad (76)$$

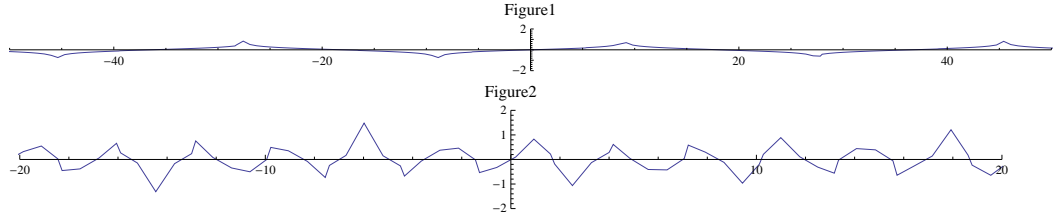
For the alternative with "+" in the expression (76) of  $x'(t)$ , we obtain that  $x'(t) > 0$ , for all  $t$ . The sign of the derivative  $z'(t)$  from (76) depends on the sign of the periodic Jacobian elliptic function  $\operatorname{cn}$ . Thus, we get, for example,

$$\begin{aligned} x'(t) > 0, \quad z'(t) < 0 & \quad \text{for } \operatorname{cn} \left( \sqrt{c_2 + A^2} t; \frac{c_2 - A^2}{c_2 + A^2} \right) < 0 \\ x'(t) > 0, \quad z'(t) > 0 & \quad \text{for } \operatorname{cn} \left( \sqrt{c_2 + A^2} t; \frac{c_2 - A^2}{c_2 + A^2} \right) > 0 \end{aligned} \quad (77)$$

In this case, is the particle trajectory (68) an undulating curve to the right? We observe that for that  $t$ 's, denoted  $\tilde{t} + K$ , with  $K$  a period, for which the periodic Jacobian elliptic function  $\text{sn}\left(\sqrt{c_2 + A^2} t; \frac{c_2 - A^2}{c_2 + A^2}\right) = \pm 1$ , we have

$$\lim_{t \rightarrow (\tilde{t} + K)} x(t) = \text{finite} := \tilde{x} + K, \quad \lim_{t \rightarrow (\tilde{t} + K)} z(t) = \pm \infty \quad (78)$$

Therefore, at  $x = \tilde{x} + K$  the graph of the curve (68) will be asymptotic. Using Mathematica, for example, for  $\delta = 1$ ,  $c = 10$ , by (27) we get  $A^2 = 1.08704$ , and choosing  $c_1 = 7.91296 > A^2$ ,  $c_2 = 2.91296 > A^2$ , the graph of the curve (68) with "+" in the expressions of  $x(t)$  and  $z(t)$ , is drawn in Figure 1. For  $\delta = \frac{1}{2}$ ,  $c = 10$ , from (27) we get  $A^2 = 146.07$ , and with  $c_1 = 177.93 > A^2$ ,  $c_2 = 253.93 > A^2$ , the graph of the curve (68) with "+" in the expressions of  $x(t)$  and  $z(t)$ , looks like in Figure 2.



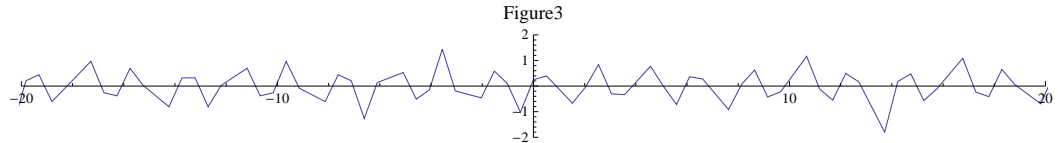
For the alternative with "-" in the expression (76) of  $x'(t)$  we obtain that

$$\begin{aligned} &\text{if } c_1 + A^2 < 4\pi^2 c^2 \text{ then } x'(t) > 0 \text{ for all } t \\ &\text{if } c_1 - A^2 > 4\pi^2 c^2 \text{ then } x'(t) < 0 \text{ for all } t \\ &\text{if } c_1 - A^2 < 4\pi^2 c^2 < c_1 + A^2 \text{ then} \\ &\quad x'(t) < 0 \text{ for } \left| \text{sn}\left(\sqrt{c_1 + A^2} t; \frac{2A^2}{c_1 + A^2}\right) \right| < \sqrt{\frac{c_1 + A^2 - 4\pi^2 c^2}{2A^2}} \\ &\quad x'(t) > 0 \text{ for } \left| \text{sn}\left(\sqrt{c_1 + A^2} t; \frac{2A^2}{c_1 + A^2}\right) \right| > \sqrt{\frac{c_1 + A^2 - 4\pi^2 c^2}{2A^2}} \end{aligned} \quad (79)$$

The sign of the derivative  $z'(t)$  from (76) depends on the sign of the periodic Jacobian elliptic function  $\text{cn}$ ; that is, the sign of the derivative  $z'(t)$  alternates successively,  $z'(t) < 0$  and  $z'(t) > 0$ .

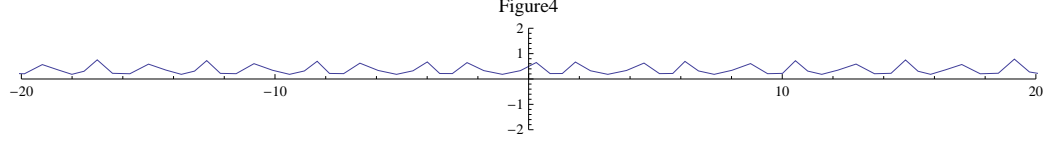
In this case, is the particle trajectory (68) an undulating curve to the right, or an undulating curve to the left, or a looping curve?

Using Mathematica, for  $\delta = \frac{1}{2}$ ,  $c = 10$ ,  $A^2 = 146.07$ ,  $c_1 = 3822.93 > A^2$  which satisfies  $c_1 - A^2 < 4\pi^2 c^2 < c_1 + A^2$  and  $c_2 = 2353.93 > A^2$ , the graph of the curve (68) with "-" in the expression of  $x(t)$  and "+" in the expression of  $z(t)$ , is drawn in Figure 3.



Using Mathematica, one can also draw the other solutions (69)-(73). For example, for  $\delta = \frac{1}{2}$ ,  $c = 10$ ,  $A^2 = 146.07$ ,  $-A^2 < c_1 = 46.07 < A^2$  and

$c_2 = -253.93 < -A^2$ , the graph of the curve (73) with "+" in the expressions of  $x(t)$  and  $z(t)$ , is drawn in the figure below.



We remark that the curves obtained are not closed curves.

## II) $c_0 \neq c$

Differentiating system (25) with respect to  $t$ , we get

$$\frac{d^2 X}{dt^2} + b \tan(X) \frac{dX}{dt} + A^2 \sin(2X) - b^2 \tan(X) = 0 \quad (80)$$

where  $A^2$  is the constant from (27) and

$$b := 2\pi(c_0 - c) \quad (81)$$

Using the substitution (31), the equation (80) takes the form

$$\frac{d^2 y}{dt^2} - \frac{2y}{1+y^2} \left( \frac{dy}{dt} \right)^2 + by \frac{dy}{dt} + 2A^2 y - b^2 y(1+y^2) = 0 \quad (82)$$

This differential equation can be written as an Abel differential equation of the second kind (see [30]). It is solvable and its solution has the parametric form (for more details, see [30]):

$$y(\tau) = \pm \sqrt{\frac{\tau^2 - 2A^2}{(C - b \ln |\tau + \sqrt{\tau^2 - 2A^2}|)^2} - 1}, \quad (83)$$

$C$  is a constant, and the relation between  $t$  and  $\tau$  is the following:

$$t = \int \frac{1}{\sqrt{\tau^2 - 2A^2} \sqrt{\tau^2 - 2A^2 - (C - b \ln |\tau + \sqrt{\tau^2 - 2A^2}|)^2}} d\tau \quad (84)$$

Thus, taking into account (31), we obtain

$$X(t) = \arctan [y(t)], \quad (85)$$

with  $y(\tau)$  given by (83) and  $\tau$  given implicitly by (84).

In order to determine  $Z(t)$  from system (25), with (85) in view, we write the second equation of this system in the form

$$\frac{dZ}{\sinh(Z)} = \frac{4\pi^2 \delta c}{\sinh(2\pi \delta)} \sin(\arctan [y(t)]) dt = \frac{4\pi^2 \delta c}{\sinh(2\pi \delta)} \frac{y(t)}{\sqrt{1+y^2(t)}} dt \quad (86)$$

If

$$\int \frac{4\pi^2 \delta c}{\sinh(2\pi \delta)} \frac{y(t)}{\sqrt{1+y^2(t)}} dt + \text{const} < 0 \quad (87)$$

then we get (see [30])

$$Z(t) = 2\operatorname{arctanh} \left[ \exp \left( \int \frac{4\pi^2\delta c}{\sinh(2\pi\delta)} \frac{y(t)}{\sqrt{1+y^2(t)}} dt + \operatorname{const} \right) \right] \quad (88)$$

From (24), (85) and (88), the solution of system (22) is written now as

$$\begin{aligned} x(t) &= ct + \frac{1}{2\pi} \operatorname{arctan} [y(t)] \\ z(t) &= \frac{1}{\pi\delta} \operatorname{arctanh} \left[ \exp \left( \int \frac{4\pi^2\delta c}{\sinh(2\pi\delta)} \frac{y(t)}{\sqrt{1+y^2(t)}} dt + \operatorname{const} \right) \right] \end{aligned} \quad (89)$$

with  $y(\tau)$  given by (83) and  $\tau$  given implicitly by (84). Taking into account (83), (84) and (27), we get

$$\int \frac{4\pi^2\delta c}{\sinh(2\pi\delta)} \frac{y(t)}{\sqrt{1+y^2(t)}} dt = \frac{1}{2} \log \left| \frac{\tau(t) - \sqrt{2}A}{\tau(t) + \sqrt{2}A} \right| \quad (90)$$

Thus, the solution in (89) has, as function of the parameter  $\tau$ , the following form:

$$\begin{aligned} x(\tau) &= c \int \frac{1}{\sqrt{\tau^2 - 2A^2} \sqrt{\tau^2 - 2A^2 - (C - b \ln |\tau + \sqrt{\tau^2 - 2A^2}|)^2}} d\tau \\ &\quad \pm \frac{1}{2\pi} \operatorname{arctan} \left[ \sqrt{\frac{\tau^2 - 2A^2}{(C - b \ln |\tau + \sqrt{\tau^2 - 2A^2}|)^2} - 1} \right] \\ z(\tau) &= \pm \frac{\operatorname{const}}{\pi\delta} \operatorname{arctanh} \left[ \sqrt{\frac{\tau - \sqrt{2}A}{\tau + \sqrt{2}A}} \right] \end{aligned} \quad (91)$$

We also remark that the above curve is not a closed curve.

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